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Inventiones mathematicae

Whitehead$_3$ is a "Slice" Link *

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We show that the Whitehead double[1] $Wh(L)$ of a two component link L is (topologically flat) slice if and only if the linking number of L is zero. When they exist, the slices may be chosen so that the complement (B^4-slices) is homotopy equivalent to a wedge of two circles, $S^1 \vee S^1$, with certain meridinal loops of $Wh(L)$ freely generating $\pi_1(B^4$-slices). (Compare this with [F1] where it is shown that the Whitehead double of boundary links are slice.) This is consistent with the surgery conjecture but seems not to illuminate the key 3-component case (see [F2]). In particular $Wh_3 = Wh(Wh_1)$ is slice. We will describe the argument quite explicitly for this case and indicate in closing remarks what modifications are needed for the general result.

First we deal with non-existence. Let S be the obvious two component Seifert surface for $Wh(L)$.

A component of S is obtained by composing either of the two punctured tori $T^2_- \subset S^1 \times D^2$ pictured below with an untwisted identification of $S^1 \times D^2$ with a neighborhood of a component of L.

Add a band to S, $\bar{S} = S \cup$ band, so that $\bar{S}$ becomes the Seifert surface for some knot, K. The Seifert forms for S and $\bar{S}$ are identical and in a natural basis will be:

$$\begin{vmatrix} 0 & a & n & n \\ b & 0 & n & n \\ n & n & 0 & c \\ n & n & d & 0 \end{vmatrix}$$

where n = the linking number of the two components and $\{a, b\} = \{0, 1\}$ or $\{0, -1\}$ and $\{c, d\} = \{0, 1\}$ or $\{0, -1\}$. Computing shows that the absolute value of the determinant of the symmetrized matrix is $|16n^2 \pm 1|$ in all cases. Clearly $|16n^2 \pm 1|$ is never a square unless $n = 0$. On the other hand this number is the order of $H_1(B; Z)$, where B is the two-fold branched cover of S^3 along

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[1] Strictly Whitehead doubling is not unique; there will be clasp choices. Hence one should read "some" or "all" for "the". Both lead to correct assertions

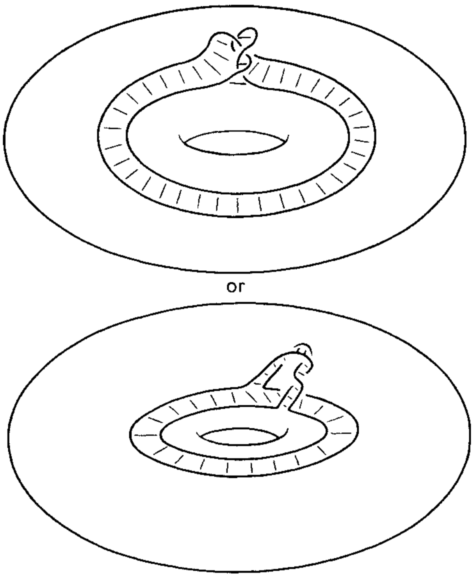


Fig. 0

K, which must be a square (see Lemma 4.2 and 4.3 [CG]) if K is slice. It follows that for $n \neq 0$, K and hence $Wh(L)$ is not slice.

We use the notation of [F1] and as in that paper proceed to construct a suitable closed slice complement M in the case $n=0$. Thus $Wh(L)$ will be sliced by abstractly building – by surgery – a 4-manifold with appropriate boundary [$\partial M = S(Wh(L))$ = the three-manifold obtained by zero-framed surgery on $Wh(L)$] and appropriate homotopy type. We cannot simply quote a surgery theorem such as appears in [F2] since the relevant fundamental group, Free (x, y) has exponential growth.

First we consider the case of $Wh_3 = Wh$ (Whitehead link). This allows the reader to follow the constructions concretely. The modifications necessary for the general case are specified at the end. The strategy for producing the closed slice complement is to produce an $S^2 \times S^2$-stabilized version in which the second homology (or surgery kernel) is sufficiently isolate from the fundamental group. This insulation of the kernel from π_1 is achieved by the construction of a suitable bordism Q on $S(Wh(L))$.

Form $W = S(L) \times [0, 1] \bigcup_{\text{meridian}_1 \times 1} (\text{kinky handle}_1) \bigcup_{\text{meridian}_2 \times 1} (\text{kinky handle}_2)$.

Let $W^+ = W \bigcup_{0\text{-framing}}$ 2-handle, the 2 handle being attached to $\hat{r} \times 0$. Set S^2 to be the "visible" immersed 2-sphere Σ dual to the extended core k' of the first kinky handle. To elaborate, $k' \subset W$ is the core of the first kinky handle union $m_1 \times [0, 1]$. Picture the Seifert surface σ for the long component of Wh_1 (as drawn in Fig. 1) – use a punctured torus as sketched in Fig. 0 – and push off a parallel copy.

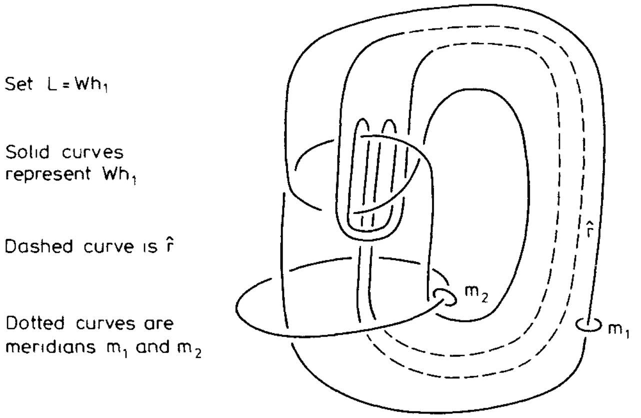


Fig. 1

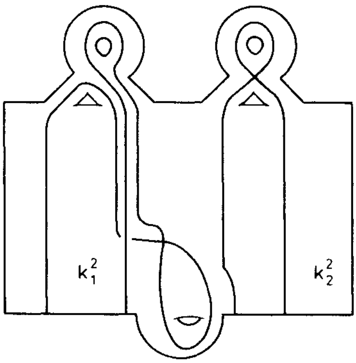


Fig. 2

The boundary of this parallel copy is almost $\hat{r}$; to obtain $\hat{r}$ simply untwist one of the bands of the copy. Inside $S(Wh_1) \times [0, \varepsilon]$, there is a singular punctured disk with boundary equal to $\hat{r}$ and the puncture limiting to the (untwisted) longitude of the "long curve" of Wh_1. In the surgered manifold $S(Wh_1)$ this map completes to a singular disk whose boundary is capped off by the core of the 2-handle attached to $\hat{r}$. The result, singular disk $\cup$ cap is an immersed sphere contained in W^+ with one double point and one (transverse) intersection with $m_1 \times [0, 1]$. As in Fig. 6 [F1] perform a singular Norman trick to define $k_1^2 \subset W_1^+$. Specifically k_1^2 will be a thickening of the singular 2-dimensional disk obtained from $k' \cup \Sigma$ by deleting a small disk in Σ about $\Sigma \cap k'$ and a small sub-disk of k' at one sheet of its double point and then connecting what remains by an imbedded cylinder. Like k' the new disk has one double point, however, it represents an improvement since the loop associated to this double point is now trivial in $\pi_1(W^+)$. k_2^2 will simply be a thickening of the extended core of the second kinky handle.

Define $Q = W^+$-interior $(k_1^2 \cup k_2^2)$. Q does not satisfy all the properties of Lemma 3 [F1] but it will play a similar role. It is an intermediate in the construction of the closed slice complement. The boundary of Q is in two connected components; the "upper" boundary $\partial^+ Q$ is obtained from $S(Wh)$ by 0-framed kinky surgery on meridians: $\partial^+ Q = S(Wh(Wh_1)) = S(Wh_3)$.

The kinky handle k_1^2 has one double point whose sign depends on the sign of the clasp in Fig. 1. The homological framing of the attaching map of k_1^2 is: framing $(k') + 2\Sigma \cdot k' + \Sigma \cdot \Sigma = 0 \pm 2 + 0$. Thus $\partial^- Q$ is given by the link diagram:

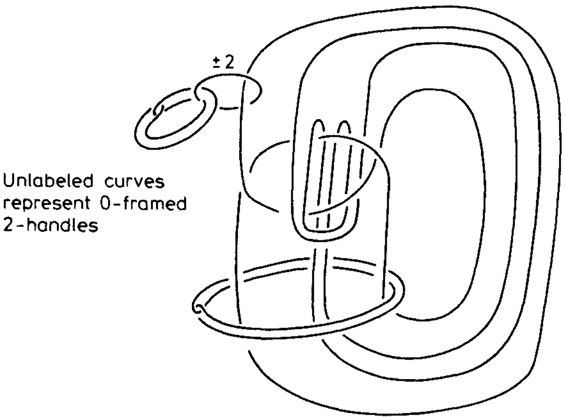


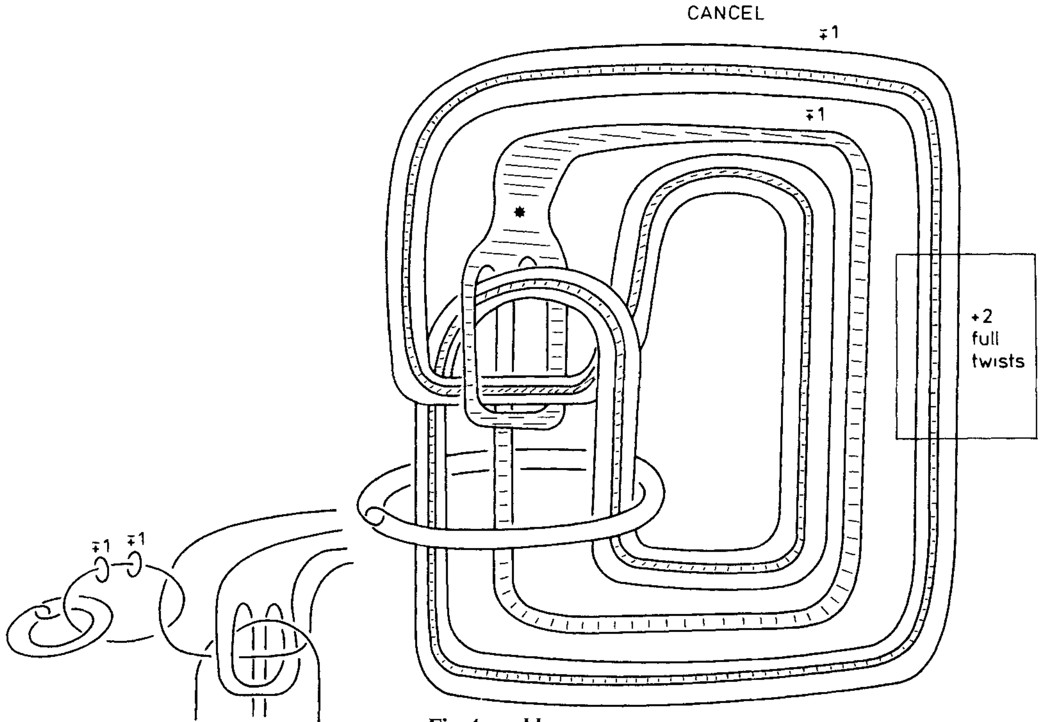
Fig. 3

Simplify as follows:

The zero framed components in Fig. 3b are a good-∂-link as defined in [F1][2], so S (0-framed components) is $Z[\pi]$-equivalent to $\underset{3}{\#} S^1 \times S^2$. To help check this, Seifert surfaces have been partially shaded in Fig. 4b, above. For both curves of Fig. 4b with framing ± 1 the various lifts to the free cover of S (0-framed components of figure) have all linking numbers zero and hence framings equal to ± 1. (The free cover is determined by the three Seifert surfaces visible in Fig. 3.) This is apparent since the ± 1 framed curves bound disjoint oriented surfaces (parallel copies of σ) in the complement of the three-surfaces determining the cover. (In the general case, the treatment of the ± 1 framed curve must be slightly different.)

Thus the free cover (again determined by the Seifert surfaces in Fig. 4b) of $\partial^- Q$ also has the integral homology of (S^3-Cantor set). It follows ([F1]) that there exists a manifold P with $\partial^- Q = \partial P$ where $P \simeq \underset{3}{\vee} S^1 \vee (\underset{n}{\vee} (S^2 \vee S^2))$ and a collection of meridians to the 0-framed components is a free basis for $\pi_1(P)$ and the homology group $H_2(P; Z[\pi])$ is a direct sum of standard planes

[2] Expressing a link as an (oriented) boundary link determines an epimorphism $\pi_1(S^3 - L) \to$ Free (# of components). If the kernel of this map is a perfect group we say that L is expressed as a "good boundary link"



a b

Fig. 4a and b

each represented by a disjoint capped grope $-S^2 \vee S^2$ s. Let $\{K\text{'s}\}$ denote the collection of these.

Form $V = Q \bigcup\limits_{\partial^- Q \equiv \partial P} P$.

Claim 1. $\pi_1(V)$ is freely generated by a collection of meridians to $Wh(L) = Wh_3$.

Claim 2. The image of $\pi_1(P)$ in $\pi_1(V)$ is isomorphic to the integers, Z, and generated by the second kinky handle.

Claim 3. $H_2(V; Z(\pi_1(V)))$ is a direct sum of standard planes generated by the lifts of $\{K\text{'s}\}$.

Since the height raising procedure for capped gropes operates in a regular neighborhood of $\{K\text{'s}\}$, Claim 2 allows the Z-theory [F2] to be applied to compress the group elements carried by the K's into $\ker(\pi_1 P \to Z)$. These elements are then trivial in $\pi_1(V)$. Thus the upper stages of each K may be replaced with disjoint twice capped gropes in V which are known to contain topological 2-handles. The result is to realize H_2 above by disjoint capped gropes with the caps topologically-flat-imbedded and 0-framed. After contracting, each standard plane summand is realized by a disjoint imbedding of a product of 2-spheres minus the interior of a flat cell $S^2 \times S^2 - D^4$. Cutting these out and gluing in 4-cells results in $M^4 = V/\text{surgery}$. The boundaries $\partial M^4 = \partial V = \partial^+ Q$ are identical

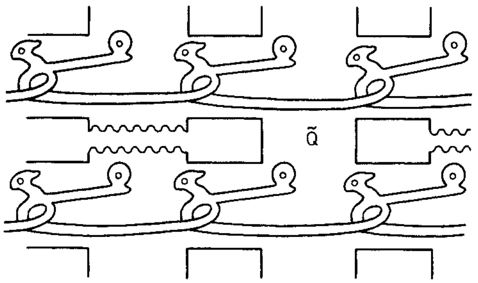


Fig. 5

and Claim 1 now applies with V replaced by M. The homology group $H_2(M; Z[\pi])$ vanishes, so by duality $\tilde{M}^{\text{univ.}}$ is contractable. Thus $M \simeq S^1 \vee S^1$ with meridians of $Wh(L)$ freely generating $\pi_1(M)$. The 4-manifold M is the desired closed slice complement for Wh_3. It remains to justify the three claims.

Claim 1. The space Q may have a rather complicated fundamental group but $\pi_1(Q)/\langle l_1, l_2\rangle = \text{Free (meridians of } Wh(L)) = F$, where l_i is a small linking circle to K_i^2, $i=1$ or 2. Since $\pi_1 \partial^- Q \to \pi_1 P$ is a epimorphism and since the loops l_i are null homotopic in P. (This is because the l_i are disjoint from the solid tori in $\partial^- Q$ which dually generate $\pi_1(P)$.) $\pi_1(V)$ is a quotient of F. In fact $\pi_1(V) \cong F$. This may be seen in several ways, e.g., Stalling's theorem [S] relating group homology to the lower central series tells us that $\pi_1(\partial V) \to \pi_1(V)$ is an isomorphism on all nilpotent quotients. Since $\pi_1(\partial V)/(\pi_1 \partial V)_k \cong F/F_k$ (recall the kernel of $\pi_1(\partial V) \to F$ is perfect) the kernel of $\pi_1(V) \to F$ must lie in F_k for all $k>0$ but $\bigcap_{k>0} F_k = \{e\}$.

Claim 2. $\pi_1(P)$ is freely generated by the loops created by the self-plumbings of the kinky handles k_1^2 and k_2^2 plus the small linking circle to $\hat{r}$. The natural free generators of $\pi_1(W^+)$ are dual to solid tori in W^+; the corresponding generators in $\pi_1(V)$ are dual to three-manifolds obtained by replacing solid tori in k_i^2 with a codimension-1 dual inside P. Among the generators of $\pi_1(P)$ only the loop associated to k_2^2 meets this dual system in V and that loop meets its three-manifold once and transversely.

Claim 3. One begins the analysis with $\tilde{W}^{+\text{univ.}}$ which is easily understood. The homology of the induced cover $\tilde{Q}$ is determined by Alexander Lefshetz duality. Schematically the inclusion of covers is drawn below:

The induced cover $\partial^- \tilde{Q}$ of $\partial^- Q$ has predictable homology. (The only nonvanishing reduced group is H_2 which is Z-freely generated by cycles linking $H_1(\partial^- Q)$.) Since we know that $\partial^- \tilde{Q} \to \#_3 S^1 \times S^2$ is a $Z\pi_1$-equivalence, the covering $\partial^- Q \to \partial^- Q$ extends to a (unique) covering $\tilde{P} \to P$. From our description of P it follows (see [FQ]) that there is a homotopy equivalence $P \to (\#_3 S^1 \times D^3) \# (\#_n S^2 \times S^2)$ and that the reduced homology of $\tilde{P}$ is generated by the lifts of $\{K\}$. Using a Mayer-Vietories sequence one may check that the homology of $\tilde{Q}$ maps to zero in $\tilde{V}=\tilde{Q}\cup\tilde{P}$ and that $\tilde{H}_2(\tilde{V}, Z)$ is generated by the lifts of $\{K\text{'s}\}$. This completes the discussion of Wh_3.

For the general case some modifications are necessary:

1. We must show that there is a collection of loops $\{\hat{r}_i\}$ with each class r_i belonging to the normal closure of commutators of meridians with conjugates of themselves $\langle [m_j, m_j^{x_k}]\rangle$; these will replace $\hat{r}$ in the previous discussion. In the general case k_1^2 may have many self plumbings; thus

2. The possibility of multiple double points in k_1^2 will not affect the argument. As before the homological framing of k_1^2 will be ± 2 and there will be two ± 1 framed loops in the analogue of Fig. 4b. It must be shown that these do not effect the $Z[\pi_1]$-homology type of $\partial^- Q$.

Let $L^+=(l_1, l'_1, l_2)$ be the three component link formed by pushing off a parallel copy (linking $\#=0$) of the first component of L. L^+ is a "minimal link" [M] and its homotopy class is given by the single integer $\mu(1,2,3)=\mu(L^+)$. It is not difficult to compute that $\mu(L^+)=0$: homotopy L to the trivial link and notice that l'_1 cannot remain parallel to l_1 though the homotopy but at each crossing the deviation is a word of the form

$$w=aba^{-1}b^{-1}b^{-1}a^{-1}ba=(aba^{-1}b^{-1})(a^{-1}b^{-1})(bab^{-1}a^{-1})(ba).$$

Since $w\in[[\pi,\pi],\pi]$ the class of l'_1 in the Milnor link group, $[l'_1]\in\mathbf{G}$ is the same as the longitude to the component l_1 as it moves through the homotopy. This is clearly trivial at the end of the homotopy so $\mu(L^+)=\mu$ (unlink)$=0$. (An alternative argument using the triple point interpretation [C] of $\mu(1,2,3)$ can likewise be given.) It follows that L^+ is null homotopic. In particular, the based homotopy class of l'_1 in S^3-L is a product $\Pi[m_{j_i}, m_{j_i}^{x_i}]$. Represent the factors by a boundary link $\{\hat{r}_i\}$, each component of which has the form shown in Fig. 2 of [F1]. Without reproducing that figure, it will suffice to construct $\hat{r}_i$ as the boundary of a surface T_i (a punctured torus) made by plumbing two annuli together. Each annuli simply links the j_i^{th} component of L (and thus carries m_{j_i}); they are plumbed along an arc whose homotopy class in S^3-L is x_i. It is important that the diagonal class in $H_1(T_i; Z)$ be untwisted – to achieve this one of the initial annuli is imbedded with a full twist and the other is untwisted. The Seifert surfaces $\{T_i\}$ should be arranged not to link each other. The $\{T_i\}$ play a role in the general case analogous to that of the single surface in Fig. 4b labeled *. Form W^+ by attaching 0-framed handles to $\{\hat{r}_i\}$. As in our example Wh_3 the purpose of attaching handles to the $\{\hat{r}_i\}$ is to create a dual sphere to the tube $M_1\times[0,1]$, and therefore to the disk k'. This is done by converting k' to k_1^2 (by the singular Norman trick) and proceeding to form Q by deleting k_1^2. The constraint on $\{\hat{r}_i\}$ is that $\partial^- Q$ must by $Z[\pi]$-homology equivalent to $\underset{\text{copies}}{\#} S^1\times S^2$. As in the case of Wh_3 the 0-framed curves in the analog of Fig. 4b constitute a good-∂-link. (Here is where the untwistedness of the diagonal class is used.)

We must now consider the effect of the ± 1-framed curves. By our construction of T_i the $\pm$ framed curves will have non-zero linking numbers with some elements of $H_1(T_i; Z)$. Let Y represent the union of the Seifert surfaces for the 0-framed components in the analog of Fig. 4b. There is a 1/2-symplectic basis β of $H_1(Y; Z)$ on which the linking pairing vanishes identically (establishing the good ∂ link property). The two ± 1 framed curves have linking number $=0$ with every curve on $Y - \bigcup_i T_i$ and also with the diagonal curves on the T_i. Thus if the three sphere S^3 is transformed by surgery on the two ± 1 framed curves (their linking number is zero) into an integral homology 3-sphere Σ^3 the link $\partial(S) \subset \Sigma^3$ is still a good-boundary-link. Consequently $\partial^- Q$ is $Z[\pi_1]$-homology-equivalent to $\underset{\text{copies}}{\#} S^1 \times S^2$.

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